Indian Statistical Institute, Bangalore
M. Math.
Second Year, Second Semester
Advanced Functional Analysis

Back paper Examination | Date : June 30, 2021
Maximum marks: 100 | Time: 3 hours
Instructor: B V Rajarama Bhat

Notation: In the following $\mathcal{H}$ is a complex separable Hilbert space and $B(\mathcal{H})$ denotes the algebra of all bounded operators on $\mathcal{H}$.

(1) Fix $n \geq 1$. Let $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$ be vectors in $\mathcal{H}$. Define $\phi : B(\mathcal{H}) \to \mathbb{C}$ by

$$\phi(X) = \sum_{j=1}^{n} \langle u_j, X v_j \rangle.$$

Prove or disprove the following claims:
(i) ϕ is continuous in SOT.
(ii) ϕ is continuous in WOT.
[15]

(2) Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis for $\mathcal{H}$. Let $S : \mathcal{H} \to \mathcal{H}$ be the unilateral shift defined by $Se_n = e_{n+1}, n \in \mathbb{N}$, and extended linearly and continuously. Take $T = 2I - S - S^*$.
(i) Show that T is a positive operator.
(ii) Let E be the spectral measure of T. Compute first three moments of the probability measure E_{e_1,e_1}. [15]

(3) Obtain polar decomposition of operators S and S^* of Question 2. [15]

(4) Fix $n \geq 1$. Let $\mathcal{U}$ be the algebra of all $n \times n$ upper-triangular matrices considered as a subalgebra M_n of all $n \times n$ complex matrices. Determine the commutant and the double commutant of $\mathcal{U}$. [15]

(5) Let A be a self-adjoint operator on a von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$.
(i) Show that there exist two positive operators A_+, A_- in $\mathcal{M}$ such that $A = A_+ - A_+$ and $A_+.A_- = A_-.A_+ = 0$.
(ii) Show that there exists a projection P in $\mathcal{M}$ such that $A_+ = AP = PA$ and $A_- = A(I - P) = (I - P)A$. [15]

(6) Let E, F are projections in a von Neumann algebra $\mathcal{A}$. Show that (i) $(E \vee F - F) \sim E - E \wedge F$; (ii) $(E - E \wedge F^{\perp}) \sim (F - E^{\perp} \wedge F)$. [15]

(7) Let U be the permutation matrix defined by

$$U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Let $\mathcal{B}$ be the unital C^*-algebra generated by U. Determine suitable compact Hausdorff space X such that $\mathcal{B}$ is isomorphic to the space $C(X)$ of continuous functions on X. Determine suitable measure space $(\Omega, \mathcal{F}, \mu)$ such that $\mathcal{B}$ is isomorphic to $L^{\infty}(\Omega, \mathcal{F}, \mu)$. [15]